Elo Rating as Relativistic Rapidity: A Mathematical Physics Perspective

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Abstract

The Elo rating system, a logistic model for ranking players in competitive games, is shown to possess a hidden mathematical structure akin to relativistic rapidity in special relativity. By expressing the expected score as a hyperbolic tangent of scaled rating differences, we derive a quantity V whose transitivity obeys Einstein's velocity addition law. This formal analogy reveals a deep connection between a probabilistic ranking system and the hyperbolic geometry of relativistic kinematics, offering a novel mathematical reinterpretation of Elo ratings.

1 Introduction

The Elo rating system, introduced by Arpad Elo [1], assigns numerical ratings to players based on pairwise competition outcomes, with the expected probability of victory modeled by a logistic function. In this note, we uncover a formal mathematical analogy: the difference in Elo ratings can be interpreted as a rapidity-like parameter from special relativity, transforming under a law isomorphic to Einstein's velocity addition.

In special relativity, rapidity $\phi = \tanh^{-1}(v/c)$ parameterizes velocity v relative to the speed of light c, and its additivity under composition reflects the hyperbolic nature of Lorentz transformations. Here, we demonstrate that the Elo expected score, when recast as $V = \tanh\left(\frac{R_A - R_B}{2k}\right)$, exhibits identical transitive properties, suggesting a structural parallel between competitive ranking and relativistic kinematics. This paper derives this connection rigorously, emphasizing its mathematical foundation.

2 Elo Rating and Expected Score

Consider two players A and B with Elo ratings R_A and R_B. The expected probability that A defeats B is:

$$P(A \text{ beats } B) = \frac{1}{1 + 10^{(R_B - R_A)/400}}$$

Defining $k = 400/\ln(10) \approx 173.7178$, this becomes:

$$P(A \text{ beats } B) = \frac{1}{1 + e^{(R_B - R_A)/k}}$$

For a scoring system awarding $+1$ for a win and -1 for a loss, the expected score V is:

$$V = 2 \cdot P(A \text{ beats } B) - 1 = \frac{1 - e^{(R_B - R_A)/k}}{1 + e^{(R_B - R_A)/k}}$$

3 Rapidity and Hyperbolic Tangent

Let $x = (R_A - R_B)/(2k)$. Then:

$$V = \frac{1 - e^{-2x}}{1 + e^{-2x}}$$

Multiplying numerator and denominator by e^x:

$$V = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \tanh(x)$$

Thus:

$$V = \tanh\left(\frac{R_A - R_B}{2 \cdot 173.7178}\right)$$

Here, $V \in [-1, 1]$ resembles a normalized velocity, with $(R_A - R_B)/(2k)$ playing the role of rapidity ϕ, satisfying $V = \tanh(\phi)$.

4 Transitivity and Einstein's Velocity Addition

For three players A, B, and C with ratings R_A, R_B, and R_C, define:

$$V_{AB} = \tanh\left(\frac{R_A - R_B}{2k}\right), \quad V_{BC} = \tanh\left(\frac{R_B - R_C}{2k}\right), \quad V_{AC} = \tanh\left(\frac{R_A - R_C}{2k}\right)$$

Given $R_A - R_C = (R_A - R_B) + (R_B - R_C)$:

$$\frac{R_A - R_C}{2k} = \frac{R_A - R_B}{2k} + \frac{R_B - R_C}{2k}$$

The hyperbolic tangent identity states:

$$\tanh(a + b) = \frac{\tanh(a) + \tanh(b)}{1 + \tanh(a)\tanh(b)}$$

Thus:

$$V_{AC} = \tanh\left(\frac{R_A - R_B}{2k} + \frac{R_B - R_C}{2k}\right) = \frac{V_{AB} + V_{BC}}{1 + V_{AB}V_{BC}}$$

This matches Einstein's velocity addition law [2]:

$$v = \frac{v_1 + v_2}{1 + v_1 v_2/c^2}$$

with $c = 1$ in normalized units, establishing V as a relativistic-like quantity.

5 Discussion

The parameter $2k \approx 347.4356$ acts as a scaling factor, mapping rating differences into a hyperbolic framework where V is bounded by ± 1, analogous to a velocity limit. This structure hints at a geometric interpretation of Elo ratings within a hyperbolic space, warranting further mathematical investigation into ranking systems.

6 Conclusion

We have demonstrated that the Elo expected score V is mathematically equivalent to a relativistic velocity, with rating differences scaled as rapidity and transitivity governed by Einstein's addition law. This reformulation positions the Elo system as a compelling example of hyperbolic geometry in applied mathematics, bridging probability and theoretical physics.

References

[1] Elo, A. E. (1978). *The Rating of Chessplayers, Past and Present.* Arco Publishing.

[2] Einstein, A. (1905). "Zur Elektrodynamik bewegter Körper" ("*On the Electrodynamics of Moving Bodies*"). *Annalen der Physik*, 322(10), 891–921.